

ON LOCAL NEWFORMS FOR UNRAMIFIED $U(2, 1)$

MICHITAKA MIYAUCHI

ABSTRACT. Let G be the unramified unitary group in three variables defined over a p -adic field F with $p \neq 2$. In this paper, we investigate local newforms for irreducible admissible representations of G . We introduce a family of open compact subgroups $\{K_n\}_{n \geq 0}$ of G to define the local newforms for representations of G as the K_n -fixed vectors. We prove the existence of local newforms for generic representations and the multiplicity one property of the local newforms for admissible representations.

INTRODUCTION

In the modern theory of automorphic representations, local newforms play a very important role. In fact, for the study of special values of automorphic L -functions through their integral presentations, the existence of local newforms is crucial. Besides the global application, local newforms are indispensable for the ramification theory of representations of p -adic groups. It was Casselman who established the local newform theory for $GL(2)$, which can be stated as follows. For a non-archimedean local field F of characteristic zero with ring of integers \mathfrak{o}_F and its maximal ideal \mathfrak{p}_F , the local counterpart of level subgroup of $GL_2(F)$ is defined by

$$\Gamma_0(\mathfrak{p}_F^n) = \left(\begin{array}{cc} \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F^n & 1 + \mathfrak{p}_F^n \end{array} \right)^\times,$$

for $n \geq 0$. For each irreducible admissible representation (π, V) of $GL_2(F)$, define the subspace

$$V(n) = \{v \in V \mid \pi(k)v = v, k \in \Gamma_0(\mathfrak{p}_F^n)\}, \quad n \geq 0.$$

Then the following was shown by Casselman [3].

Theorem 0.1 (Local newforms for $GL(2)$). *Let (π, V) be an irreducible generic representation of $GL_2(F)$.*

- (i) *There is a non-negative integer n such that $V(n) \neq \{0\}$.*
- (ii) *Put $c(\pi) = \min\{n \mid V(n) \neq \{0\}\}$. Then $\dim V(c(\pi)) = 1$.*
- (iii) *For each $n \geq c(\pi)$, we have*

$$\dim V(n) = n - c(\pi) + 1.$$

(iv) *For any non-zero element v in $V(c(\pi))$, the corresponding Whittaker function W_v to v is non-zero at 1.*

The integer $c(\pi)$ is called *the conductor of π* . It is also known that the ε -factor for π is a constant multiple of $q^{-sc(\pi)}$, where q is the cardinality of the residue class field of F . Similar results were obtained by Jacquet, Piatetski-Shapiro and Shalika [4] and by Reeder [8] for $GL_n(F)$. Recently, Roberts and Schmidt [9] established a theory of local newforms for irreducible representations of $GSp_4(F)$ with trivial central character. They considered paramodular subgroups of $GSp_4(F)$ to define local newforms. Our main concern is to construct a similar theory for unramified unitary group $U(2, 1)$. We note that for unitary group $U(1, 1)$, there is a result by Lansky and Raghuram [5], which determined the dimensions of the spaces of vectors fixed by

2010 *Mathematics Subject Classification*. Primary 22E50, 22E35.

Key words and phrases. p -adic group, local newform.

certain open compact subgroups. Unfortunately they do not concern the relation between their conductors and the exponents of ε -factors.

We assume that the residual characteristic of F is odd. Let E be the unramified quadratic extension over F . Let \mathfrak{o}_E denote the ring of integers in E , \mathfrak{p}_E the maximal ideal in \mathfrak{o}_E . We realize our group G as $\{g \in \mathrm{GL}_3(E) \mid {}^t \bar{g} J g = J\}$ and denote it by $\mathrm{U}(2, 1)(E/F)$, where $\bar{}$ is the non-trivial element in $\mathrm{Gal}(E/F)$ and $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. We define open compact subgroups K_n of G as an analog of paramodular subgroups of $\mathrm{GSp}_4(F)$;

$$(0.2) \quad K_n = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-n} \\ \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^n & \mathfrak{o}_E \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & \mathfrak{o}_E \end{pmatrix} \cap G,$$

for $n \geq 0$. We show the followings as the main results of this paper.

Theorem 0.3 (Multiplicity one of newforms). *Let (π, V) be an irreducible generic representation of G . We denote by $V(n)$ the space of K_n -fixed vectors in V .*

- (i) *There is a non-negative integer n such that $V(n) \neq \{0\}$.*
- (ii) *Put $N_\pi = \min\{n \mid V(n) \neq \{0\}\}$. Then $\dim V(N_\pi) = 1$.*

We call $V(N_\pi)$ the space of *newforms*, and $V(n)$ the space of *oldforms*, for $n > N_\pi$. The part (ii) of Theorem 0.3 can be shown for irreducible admissible representations with non-zero K_n -fixed vectors. For the uniqueness of the newforms, we do not assume any condition on the central characters. In the case when the central character is trivial in the neighborhood of newforms, we can show more:

Theorem 0.4 (Dimensions of oldforms, test vectors for the Whittaker functional). *Let (π, V) be an irreducible generic representation of G . We denote by n_π the conductor of the central character of π . Suppose that $N_\pi > n_\pi$ and $N_\pi \geq 2$. Then*

- (i) *For any $n \geq N_\pi$, we have*

$$\dim V(n) = \left\lfloor \frac{n - N_\pi}{2} \right\rfloor + 1.$$

- (ii) *A non-zero element v in $V(N_\pi)$ is a test vector for the Whittaker functional, that is, $W_v(1) \neq 0$, where W_v is the Whittaker function corresponding to v .*

We summarize the contents of this paper. In section 1, we fix the basic notation for representations of the unramified unitary group in three variables. In section 2, we introduce the notion of local newforms and prove that any irreducible generic representations of G admit local newforms. In section 3, we define two level raising operators θ' and η following Roberts and Schmidt [9]. There the P_3 -theory plays an important role to estimate the dimensions of the oldforms for $\mathrm{GSp}(4)$. Here P_n is the mirabolic subgroup of $\mathrm{GL}_n(F)$. In section 4, we recall the P_2 -theory for $\mathrm{U}(2, 1)$ from Baruch [1], and consider ‘‘Kirillov model’’ for generic representations of G . In section 5, we prove our main theorem, that is, multiplicity one theorem of local newforms (Theorem 5.6). Moreover, we give the dimension formula of oldforms for generic representations of G whose conductors are different from those of their central characters (Theorem 5.8). We also show that all generic supercuspidal representations satisfy this condition.

We have not yet obtained the dimension formula of oldforms for representations whose conductors are equal to those of central characters. Although we define the conductors of generic representations of G , here we do not consider comparison of them with their ε -factors. We hope to consider these problems in the future.

Acknowledgements The author is grateful to Yoshi-hiro Ishikawa for careful reading of the first draft of this paper, and would like to thank Takuya Yamauchi and Tadashi Yamazaki for helpful discussions.

1. PRELIMINARIES

Here we realize our unramified unitary group in three variables and summarize basic notation of its subgroups and terminology of its representations, which are used in this paper. Let F be a non-archimedean local field of characteristic zero. Let \mathfrak{o}_F be the ring of integers in F , $\mathfrak{p}_F = \varpi_F \mathfrak{o}_F$ the maximal ideal in \mathfrak{o}_F and $k_F = \mathfrak{o}_F / \mathfrak{p}_F$ the residue field. We denote by $q = q_F$ the cardinality of k_F . Let $|\cdot|_F$ denote the absolute value of F normalized so that $|\varpi_F|_F = q_F^{-1}$. We use the analogous notation for any non-archimedean local field. Throughout this paper, we assume that the characteristic p of k_F is odd.

Let $E = F[\sqrt{\epsilon}]$ be the unramified quadratic extension over F , where $\epsilon \in \mathfrak{o}_F^\times \setminus (\mathfrak{o}_F^\times)^2$. Then ϖ_F is a uniformizer of E . We abbreviate $\varpi = \varpi_F$. We denote by $-$ the non-trivial element in $\text{Gal}(E/F)$. We set $G = \{g \in \text{GL}_3(E) \mid {}^t \bar{g} J g = J\}$ where

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then G is the F -points of the unramified unitary group $\text{U}(2, 1)$ over F .

Let B denote the Borel subgroup of G consisting of the upper triangular matrices in G . We denote by T the subgroup of B consisting of the diagonal matrices in G . The unipotent radical U of B is given by

$$U = \left\{ u(x, y) = \begin{pmatrix} 1 & x & y \\ 0 & 1 & -\bar{x} \\ 0 & 0 & 1 \end{pmatrix} \mid x, y \in E, y + \bar{y} + x\bar{x} = 0 \right\}.$$

We denote the opposite of U by \hat{U} ;

$$\hat{U} = \left\{ \hat{u}(x, y) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & -\bar{x} & 1 \end{pmatrix} \mid x, y \in E, y + \bar{y} + x\bar{x} = 0 \right\}.$$

We embed the group $H = \text{U}(1, 1)(E/F)$ into G as

$$H = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in G \right\}.$$

We put $B_H = B \cap H$, $\hat{U}_H = \hat{U} \cap H$,

$$U_H = U \cap H = \left\{ u(y) = \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid y \in E, y + \bar{y} = 0 \right\}$$

and

$$T_H = T \cap H = \left\{ t(a) = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \mid a \in E^\times \right\}.$$

We fix a non-trivial additive character ψ_E of E with conductor \mathfrak{o}_E , and define a character ψ of U by

$$\psi(u) = \psi_E(x), \text{ for } u = u(x, y) \in U.$$

For any irreducible admissible representation (π, V) of G , it is well-known that

$$\dim \operatorname{Hom}_U(\pi, \psi) \leq 1.$$

We say that (π, V) is *generic* if $\operatorname{Hom}_U(\pi, \psi) \neq \{0\}$. If (π, V) is an irreducible generic representation of G , then by Frobenius reciprocity, we have

$$\operatorname{Hom}_U(\pi, \psi) \simeq \operatorname{Hom}_G(\pi, \operatorname{Ind}_U^G \psi) \simeq \mathbf{C}.$$

So there exists a unique embedding of π into $\operatorname{Ind}_U^G \psi$ up to scalar. The image $\mathcal{W}(\pi, \psi)$ of V is called *the Whittaker model* of π . By a non-zero functional $l \in \operatorname{Hom}_U(\pi, \psi)$, which is called *the Whittaker functional*, we define *the Whittaker function* $W_v \in \mathcal{W}(\pi, \psi)$ associated to $v \in V$ by

$$W_v(g) = l(\pi(g)v), \quad g \in G.$$

There is an isomorphism ι between the center Z of G and the norm-one subgroup E^1 of E^\times , given by

$$\iota : E^1 \simeq Z; \lambda \mapsto \begin{pmatrix} \lambda & & \\ & \lambda & \\ & & \lambda \end{pmatrix}.$$

We set open compact subgroups of E^1 as

$$E_0^1 = E^1, \quad E_n^1 = E^1 \cap (1 + \mathfrak{p}_E^n), \quad \text{for } n \geq 1.$$

Then $\{E_n^1\}_{n \geq 0}$ gives a filtration of E^1 . For an irreducible admissible representation (π, V) of G , we denote by ω_π the central character of π . We define *the conductor* n_π of ω_π by

$$n_\pi = \min\{n \geq 0 \mid \omega_\pi|_{Z_n} = 1\},$$

where $Z_n = \iota(E_n^1)$.

2. LOCAL NEWFORMS

In this section, we introduce the notion of newforms for representations of G . A newform for an irreducible admissible representation of G is a vector which is fixed by a certain open compact subgroup of G . We prove that every irreducible generic representation of G admits a newform (Theorem 2.8).

We introduce a family of open compact subgroups $\{K_n\}_{n \in \mathbf{Z}_{\geq 0}}$ of G by

$$K_n = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-n} \\ \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^n & \mathfrak{o}_E \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & \mathfrak{o}_E \end{pmatrix} \cap G,$$

which is used to define our local newforms. We set

$$t_n = \begin{pmatrix} & \varpi^{-n} \\ & 1 \\ \varpi^n & \end{pmatrix} \in K_n.$$

The group K_0 is a good maximal compact subgroup of G . So we have the Iwasawa decomposition $G = BK_0$. Moreover, we obtain the the following decomposition for K_1 :

Lemma 2.1. $G = BK_1$.

Proof. The quotient $K_0/(1 + M_3(\mathfrak{p}_E)) \cap G$ is isomorphic to $\operatorname{U}(2, 1)(k_E/k_F)$. Using the Bruhat decomposition of $\operatorname{U}(2, 1)(k_E/k_F)$, we get

$$(2.2) \quad G = BK_0 = BW(K_0 \cap K_1),$$

where $W = \{1, t_0\}$. Since $t_0 \in Bt_1 \subset BK_1$, we obtain $G = BK_1$. \square

We define an open compact subgroup $U(\mathfrak{o}_E)$ of U by

$$U(\mathfrak{o}_E) = \begin{pmatrix} 1 & \mathfrak{o}_E & \mathfrak{o}_E \\ 0 & 1 & \mathfrak{o}_E \\ 0 & 0 & 1 \end{pmatrix} \cap G.$$

Lemma 2.3. *For $n \geq 0$, the group K_n is generated by $K_n \cap H$ and $U(\mathfrak{o}_E)$.*

Proof. Let K' denote the subgroup of G generated by $K_n \cap H$ and $U(\mathfrak{o}_E)$. We prove that K' contains K_n .

(i) Suppose that $n \geq 1$. Put

$$K'' = \begin{pmatrix} 1 + \mathfrak{p}_E^n & \mathfrak{o}_E & \mathfrak{o}_E \\ \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^n & \mathfrak{o}_E \\ \mathfrak{p}_E^{2n} & \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^n \end{pmatrix} \cap G.$$

It is easy to check that $K_n = (K_n \cap H)K''$. So it is enough to prove that $K'' \subset K'$.

The group K'' has an Iwahori decomposition $K'' = (K'' \cap \hat{U})(K'' \cap T)(K'' \cap U)$. The group K' contains $K'' \cap U = U(\mathfrak{o}_E)$. Since $t_n \in K_n \cap H$, K' contains $K'' \cap \hat{U} = t_n U(\mathfrak{o}_E) t_n$. We have $K'' \cap T = (K'' \cap T_H)(K'' \cap Z) = (K'' \cap T_H)Z_n$. Since $K'' \cap T_H \subset K_n \cap H \subset K'$, it suffices to show $Z_n \subset K'$. We note that

$$(2.4) \quad K_n \cap U, K_n \cap \hat{U} \subset K'$$

because $K_n \cap U = (K'' \cap U)(K_n \cap U_H)$ and $K_n \cap \hat{U} = (K'' \cap \hat{U})(K_n \cap \hat{U}_H)$.

We shall prove $Z_n \subset K'$. Since $Z_n \neq Z_{n+1}$, it is enough to check $Z_n \setminus Z_{n+1} \subset K'$. Let $1 + x$ be an element in $E_n^1 \setminus E_{n+1}^1$. Then there is $z \in \mathfrak{p}_E^{-n} \setminus \mathfrak{p}_E^{-1-n}$ such that $1 + x = -\bar{z}/z$. Since $z + \bar{z} = -xz \in \mathfrak{o}_F^\times$, there is an element $y \in \mathfrak{o}_E^\times$ such that $z + \bar{z} + y\bar{y} = 0$. We have

$$(2.5) \quad \text{diag}(\varpi^n z, -\bar{z}/z, \varpi^{-n} \bar{z}^{-1}) = \hat{u}(\bar{y}/z, 1/\bar{z}) u(y, z) \hat{u}(\bar{y}/\bar{z}, 1/\bar{z}) t_n.$$

By (2.4), all elements in the right-hand side in (2.5) belong to K' . So we get

$$\text{diag}(\varpi^n z, -\bar{z}/z, \varpi^{-n} \bar{z}^{-1}) \in K'.$$

Since $\varpi^n z \in \mathfrak{o}_E^\times$, we obtain $\iota(1 + x) = \iota(-\bar{z}/z) = t(-\varpi^{-n} \bar{z}/z^2) \text{diag}(\varpi^n z, -\bar{z}/z, \varpi^{-n} \bar{z}^{-1}) \in K'$. This completes the proof for $n \geq 1$.

(ii) Suppose that $n = 0$. By (2.2), we get $K_0 = (B \cap K_0)W(K_0 \cap K_1)$. We have $W = \{1, t_0\} \subset K_0 \cap H \subset K'$ and $K_0 \cap U \subset K'$. So we get $K_0 \cap \hat{U} = t_0(K_0 \cap U)t_0 \subset K'$. Since $B \cap K_0 = (K_0 \cap T)(K_0 \cap U)$ and $K_0 \cap K_1$ has an Iwahori decomposition, it suffices to prove $K_0 \cap T \subset K'$. Note that $K_0 \cap T = (K_0 \cap T_H)Z$ and $K_0 \cap T_H \subset K'$. So it is enough to prove $Z \subset K'$.

Since $Z = Z_0 \neq Z_1$, it suffices to prove $Z \setminus Z_1 \subset K'$. Let x be an element in $E^1 \setminus E_1^1$. Then there is $z \in \mathfrak{o}_E^\times$ such that $x = -\bar{z}/z$. Since $z + \bar{z} = (1 - x)z \in \mathfrak{o}_F^\times$, there is an element $y \in \mathfrak{o}_E^\times$ such that $z + \bar{z} + y\bar{y} = 0$. So we can observe $\iota(x) = \iota(-\bar{z}/z) \in K'$ as in the case when $n \geq 1$. \square

Let (π, V) be an irreducible admissible representation of G . For each non-negative integer n , we define a subspace

$$V(n) = \{v \in V \mid \pi(k)v = v, k \in K_n\}$$

of V . Since π is admissible, $V(n)$ is finite-dimensional for all $n \geq 0$.

Definition 2.6. Let (π, V) be an irreducible admissible representation of G which has K_n -fixed vectors for some $n \geq 0$. We define the conductor of π by

$$N_\pi = \min\{n \geq 0 \mid V(n) \neq \{0\}\}.$$

We call the vectors in $V(N_\pi)$ the newforms for π , and all the elements in $V(n)$ the oldforms for π , for $n > N_\pi$.

Remark 2.7. Since $Z_n = Z \cap K_n$, the central character ω_π of π is trivial on Z_n if $V(n) \neq \{0\}$. This implies

$$N_\pi \geq n_\pi.$$

The relation between these conductors plays an important role in section 5.

The following theorem states that we can define the conductors at least for generic representations of G .

Theorem 2.8. *If an irreducible admissible representation (π, V) of G is generic, then there exists a non-negative integer n such that $V(n) \neq \{0\}$.*

Proof. We regard $|\cdot|_E$ as a quasi-character of $T_H \simeq E^\times$. It follows from [1] Theorem 4.7 that $\dim \text{Hom}_H(V, \text{Ind}_{B_H}^H |\cdot|_E^s) = 1$, outside a finite number of values of q^{2s} . So we can choose $s \in \mathbf{C}$ such that $\dim \text{Hom}_H(V, \text{Ind}_{B_H}^H |\cdot|_E^s) = 1$ and $\text{Ind}_{B_H}^H |\cdot|_E^s$ is an unramified principal series representation of $H \simeq U(1, 1)(E/F)$. Thus there exists a non-zero $K_0 \cap H$ -fixed vector v in V . Take a positive integer n so that v is fixed by

$$\begin{pmatrix} 1 & \mathfrak{p}_E^n & \mathfrak{p}_E^n \\ & 1 & \mathfrak{p}_E^n \\ & & 1 \end{pmatrix} \cap G.$$

Then it follows from Lemma 2.3 that the vector

$$\pi \begin{pmatrix} \varpi^{-n} & & \\ & 1 & \\ & & \varpi^n \end{pmatrix} v$$

lies in $V(2n)$. □

3. LEVEL RAISING OPERATORS

Let (π, V) be an irreducible admissible representation of G . In this section, we define two level raising operators $\eta : V(n) \rightarrow V(n+2)$ and $\theta' : V(n) \rightarrow V(n+1)$ as in [9] subsection 3.2, and study their several properties.

We set

$$\eta = \begin{pmatrix} \varpi^{-1} & & \\ & 1 & \\ & & \varpi \end{pmatrix}.$$

For $n \geq 0$, we have $K_{n+2} \subset \eta K_n \eta^{-1}$. So we can define an operator $\eta : V(n) \rightarrow V(n+2)$ by

$$\eta v = \pi(\eta)v, \quad v \in V(n).$$

We define an open compact subgroup $U(\mathfrak{p}_E^{-1})$ of U by

$$U(\mathfrak{p}_E^{-1}) = \begin{pmatrix} 1 & \mathfrak{p}_E^{-1} & \mathfrak{p}_E^{-2} \\ 0 & 1 & \mathfrak{p}_E^{-1} \\ 0 & 0 & 1 \end{pmatrix} \cap G.$$

Lemma 3.1. *Let n be a non-negative integer and v an element in $V(n+2)$. Then $v \in \eta V(n)$ if and only if v is fixed by $U(\mathfrak{p}_E^{-1})$.*

Proof. Observe that $\eta(K_n \cap H)\eta^{-1} = K_{n+2} \cap H$ and $\eta U(\mathfrak{o}_E)\eta^{-1} = U(\mathfrak{p}_E^{-1})$. Lemma 2.3 says that the group $\eta K_n \eta^{-1}$ is generated by $K_{n+2} \cap H$ and $U(\mathfrak{p}_E^{-1})$. The assertion follows immediately from this. □

We define another level raising operator $\theta' : V(n) \rightarrow V(n+1)$ by

$$\theta'v = \frac{1}{\text{vol}(K_{n+1} \cap K_n)} \int_{K_{n+1}} \pi(k)vd k, \quad v \in V(n).$$

To describe θ' explicitly, we prepare the following:

Lemma 3.2. *Let n be a non-negative integer. Then a complete system of representatives for $K_{n+1}/K_{n+1} \cap K_n$ is given by $q+1$ elements t_{n+1} and $u(x\sqrt{\epsilon})$, $x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n}$.*

Proof. We have

$$K_{n+1} \cap K_n = \begin{pmatrix} \mathfrak{o}_E & \mathfrak{o}_E & \mathfrak{p}_E^{-n} \\ \mathfrak{p}_E^{n+1} & 1 + \mathfrak{p}_E^{n+1} & \mathfrak{o}_E \\ \mathfrak{p}_E^{n+1} & \mathfrak{p}_E^{n+1} & \mathfrak{o}_E \end{pmatrix} \cap G.$$

It is easy to observe that the elements in the assertion belong to pairwise distinct cosets in $K_{n+1}/K_{n+1} \cap K_n$. We write the (i, j) -entry of $k \in K_{n+1}$ as k_{ij} . Suppose that $k_{33} \in \mathfrak{p}_E$. Then we have $t_{n+1}k \in K_{n+1} \cap K_n$, and hence $k \in t_{n+1}(K_{n+1} \cap K_n)$. If $k_{33} \in \mathfrak{o}_E^\times$, then we have $k_{13}\bar{k}_{33} + k_{23}\bar{k}_{23} + \bar{k}_{13}k_{33} = 0$ because k lies in G . This implies $k_{13}\bar{k}_{33} \in \mathfrak{o}_F \oplus \mathfrak{p}_F^{-1-n}\sqrt{\epsilon}$. Since $k_{33}\bar{k}_{33} \in \mathfrak{o}_F^\times$, we get $k_{13}k_{33}^{-1} \in \mathfrak{o}_F \oplus \mathfrak{p}_F^{-1-n}\sqrt{\epsilon}$. So there exists $x \in \mathfrak{p}_F^{-1-n}$ such that $k_{13} - x\sqrt{\epsilon}k_{33} \in \mathfrak{o}_E$. We therefore have $k \in u(x\sqrt{\epsilon})(K_{n+1} \cap K_n)$. \square

Proposition 3.3. *Let n be a non-negative integer. Then we have*

$$\theta'v = \eta v + \sum_{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} v, \quad v \in V(n).$$

Proof. The proposition follows from Lemma 3.2 and the equation $t_{n+1} = \eta t_n$. \square

By Proposition 3.3, the operators θ' and η commute each other.

Corollary 3.4. *Let n be a non-negative integer. We have $\eta\theta'v = \theta'\eta v$, for all $v \in V(n)$.*

We prepare three Lemmas 3.5, 3.6 and 3.7, whose proofs are similar to those of Theorems 3.2.5, 3.2.6 and Lemma 3.4.1 in [9] respectively.

Lemma 3.5. *Let n be a positive integer. Let v be an element in V such that*

$$\sum_{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} v = 0.$$

Suppose that v is fixed by the following subgroups of G :

$$(i) \begin{pmatrix} 1 & \mathfrak{p}_E^{-n} \\ & 1 & \\ & & 1 \end{pmatrix} \cap G, \quad (ii) \begin{pmatrix} \mathfrak{o}_E^\times & & \\ & 1 & \\ & & \mathfrak{o}_E^\times \end{pmatrix} \cap G, \quad (iii) \begin{pmatrix} 1 & & \\ \mathfrak{p}_E^n & 1 & \\ \mathfrak{p}_E^{n+1} & \mathfrak{p}_E^n & 1 \end{pmatrix} \cap G.$$

Then v is fixed by t_{n+1} and $U(\mathfrak{p}_E^{-1})$.

Proof. Since v is fixed by the subgroup (i), the sum

$$\sum_{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} v$$

is well-defined.

We claim that v is fixed by t_{n+1} . By assumption, we have

$$-v = \sum_{\substack{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n} \\ x \neq 0}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 \\ & & 1 \end{pmatrix} v.$$

Because v is fixed by the subgroup (ii) and (iii), we obtain

$$\begin{aligned} -\pi(t_{n+1})v &= \sum_{\substack{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n} \\ x \neq 0}} \pi \left(t_{n+1} \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 \\ & & 1 \end{pmatrix} \right) v \\ &= \sum_{\substack{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n} \\ x \neq 0}} \pi \left(\begin{pmatrix} 1 & \varpi^{-2-2n}x^{-1}\sqrt{\epsilon}^{-1} \\ & 1 \\ & & 1 \end{pmatrix} \begin{pmatrix} -\varpi^{-1-n}x^{-1}\sqrt{\epsilon}^{-1} & & \\ & 1 & \\ & & \varpi^{n+1}x\sqrt{\epsilon} \end{pmatrix} \right) v \\ &= \sum_{\substack{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n} \\ x \neq 0}} \pi \begin{pmatrix} 1 & \varpi^{-2-2n}x^{-1}\sqrt{\epsilon}^{-1} \\ & 1 \\ & & 1 \end{pmatrix} v \\ &= -v. \end{aligned}$$

Therefore, v is fixed by t_{n+1} .

Since

$$U(\mathfrak{p}_E^{-1}) = t_{n+1} \left(\begin{pmatrix} 1 & & \\ \mathfrak{p}_E^n & 1 & \\ \mathfrak{p}_E^{2n} & \mathfrak{p}_E^n & 1 \end{pmatrix} \cap G \right) t_{n+1}$$

and v is fixed by the subgroup (iii), we see that v is fixed by $U(\mathfrak{p}_E^{-1})$. \square

We introduce one more operator S on V ;

$$Sv = \frac{1}{\text{vol}(U(\mathfrak{o}_E))} \int_{U(\mathfrak{p}_E^{-1})} \pi(u)vdu, \quad v \in V.$$

Let n be a non-negative integer and let $v \in V(n)$. One can observe that v is fixed by $U(\mathfrak{p}_E^{-1})$ if and only if $Sv = q^4v$. If $n \geq 2$, it follows from Lemma 3.1 that $v \in \eta V(n-2)$ if and only if $Sv = q^4v$.

Lemma 3.6. *Let n be an integer such that $n \geq 2$ and $n > n_\pi$. Suppose that an element v in $V(n)$ satisfies $\theta'v \in \eta V(n-1)$. Then $v \in \eta V(n-2)$.*

Proof. By assumption and Proposition 3.3, we have

$$\begin{aligned} 0 &= (S - q^4)\theta'v = (S - q^4)(\eta v + \sum_{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 \\ & & 1 \end{pmatrix} v) \\ &= (S - q^4) \sum_{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 \\ & & 1 \end{pmatrix} v \\ &= \sum_{x \in \mathfrak{p}_F^{-1-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 \\ & & 1 \end{pmatrix} (S - q^4)v \end{aligned}$$

because ηv is fixed by $U(\mathfrak{p}_E^{-1})$ and U_H is the center of U .

We claim that $(S - q^4)v$ is fixed by the subgroups (i)-(iii) in Lemma 3.5. Then Lemma 3.5 says that $(S - q^4)v$ is fixed by $U(\mathfrak{p}_E^{-1})$. Since Sv is fixed by $U(\mathfrak{p}_E^{-1})$, we see that v is fixed by $U(\mathfrak{p}_E^{-1})$. Therefore we get $v \in \eta V(n-2)$ by Lemma 3.1.

We shall prove the claim. Since v is fixed by the subgroups (i)-(iii) in Lemma 3.5, it is enough to check that Sv is fixed by them. It is obvious that Sv is fixed by the subgroups (i) and (ii). We shall show that Sv is fixed by the subgroup (iii). Since $U(\mathfrak{o}_E)$ normalizes $(1 + M_3(\mathfrak{p}_E^{n-1})) \cap G$, the group $U(\mathfrak{p}_E^{-1}) = \eta U(\mathfrak{o}_E) \eta^{-1}$ normalizes

$$K' = \eta(1 + M_3(\mathfrak{p}_E^{n-1}))\eta^{-1} \cap G = \begin{pmatrix} 1 + \mathfrak{p}_E^{n-1} & \mathfrak{p}_E^{n-2} & \mathfrak{p}_E^{n-3} \\ \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^{n-1} & \mathfrak{p}_E^{n-2} \\ \mathfrak{p}_E^{n+1} & \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^{n-1} \end{pmatrix} \cap G.$$

Since the subgroup (iii) lies in K' , it is enough to prove that v is fixed by K' . Note that $K' \subset Z_{n-1}K_n$. Since we are assuming $n-1 \geq n_\pi$, the group Z_{n-1} acts on V trivially. We therefore conclude that v is fixed by K' . This completes the proof. \square

Lemma 3.7. *Let n and k be non-negative integers. For $v \in V(n)$, there exist $v_1 \in V(n+k-2)$ and $v_2 \in V(n+k-1)$ such that*

$$\int_{\mathfrak{p}_F^{-k-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} v dx = \theta'^k v + \eta v_1 + \eta v_2.$$

Here, we put $v_1 = 0$ if $n+k-2 < 0$ and $v_2 = 0$ if $n+k-1 < 0$.

Proof. We shall prove the lemma by induction on k . Suppose that $k = 0$. Then the assertion is true with $v_1 = v_2 = 0$.

Suppose that $k > 0$. Then, by the induction hypothesis, we have

$$\begin{aligned} & \int_{\mathfrak{p}_F^{-k-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} v dx \\ &= \int_{\mathfrak{p}_F^{-k-n}/\mathfrak{p}_F^{1-k-n}} \pi \begin{pmatrix} 1 & y\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} \left(\int_{\mathfrak{p}_F^{1-k-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} v dx \right) dy \\ &= \int_{\mathfrak{p}_F^{-k-n}/\mathfrak{p}_F^{1-k-n}} \pi \begin{pmatrix} 1 & y\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} (\theta'^{k-1}v + \eta v'_1 + \eta v'_2) dy, \end{aligned}$$

for some $v'_1 \in V(n+k-3)$ and $v'_2 \in V(n+k-2)$. By Proposition 3.3 and the fact $\eta v'_2 \in V(n+k)$, we get

$$\begin{aligned} \int_{\mathfrak{p}_F^{-k-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} v dx &= (\theta' - \eta)(\theta'^{k-1}v + \eta v'_1) + q\eta v'_2 \\ &= \theta'^k v + \eta(\theta'v'_1 + qv'_2) + \eta(-\theta'^{k-1}v - \eta v'_1). \end{aligned}$$

Put $v_1 = \theta'v'_1 + qv'_2$ and $v_2 = -\theta'^{k-1}v - \eta v'_1$. This completes the proof. \square

4. P_2 -THEORY

In subsection 4.1, we recall P_2 -theory for $U(2,1)$ from [1] section 4, and relate this to the level raising operators (see Lemma 4.2). In subsection 4.2, we develop P_2 -theory to the “Kirillov models” for generic representations, and prove multiplicity one theorem of newforms for generic representations whose conductors differ from those of their central characters (Theorem 4.11).

4.1. **P_2 -modules.** Let P_2 be the subgroup of $\mathrm{GL}_2(E)$ consisting of the matrices of the form

$$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}.$$

We get an isomorphism $T_H U / U_H \simeq P_2$ from the map

$$t(a)u(x, y) \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad a \in E^\times, \quad x, y \in E.$$

For an irreducible admissible representation (π, V) of G , we set $V_{U_H} = V / \langle \pi(u)v - v \mid v \in V, u \in U_H \rangle$. Then we can regard V_{U_H} as a P_2 -module. We denote by p the natural projection from V to V_{U_H} .

We recall from [2] section 5 the structure of P_2 -modules. We put

$$Z_2 = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \mid x \in E \right\}$$

and

$$V_{U_H}(Z_2) = \langle \pi(z)v - v \mid v \in V_{U_H}, z \in Z_2 \rangle.$$

Then $V_{U_H}(Z_2)$ is a P_2 -submodule of V_{U_H} and $V_{U_H}/V_{U_H}(Z_2) \simeq V_U$, where V_U is the unnormalized Jacquet module of (π, V) . Let $V_{U, \psi}$ denote the twisted Jacquet module of (π, V) with respect to (U, ψ) . Then $V_{U_H}(Z_2)$ is isomorphic to $(\dim V_{U, \psi}) \cdot \mathrm{ind}_{Z_2}^{P_2}(\psi)$, where ψ is the character of Z_2 defined by

$$\psi \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} = \psi_E(x), \quad x \in E$$

and ind denotes the compactly supported induction. We note that $\mathrm{ind}_{Z_2}^{P_2}(\psi)$ is an irreducible P_2 -module.

We summarize a criterion of genericity and supercuspidality of π in terms of P_2 -modules:

Proposition 4.1. *Let (π, V) be an irreducible admissible representation of G .*

- (i) *π is supercuspidal if and only if $V_{U_H} = V_{U_H}(Z_2)$;*
- (ii) *π is generic if and only if $V_{U_H}(Z_2) \neq \{0\}$. Moreover, if this is the case, then $V_{U_H}(Z_2) \simeq \mathrm{ind}_{Z_2}^{P_2}(\psi)$.*

The following lemma gives a criterion whether or not, a K_n -fixed vector comes via the operator η , which is our main tool to prove the uniqueness of newform.

Lemma 4.2. *Let (π, V) be an irreducible admissible representation of G and let n be an integer such that $n \geq 2$ and $n > n_\pi$. Suppose that an element v in $V(n)$ satisfies $p((S - q^4)v) = 0$. Then v belongs to $\eta V(n - 2)$.*

Proof. The assumption implies $(S - q^4)v \in \langle \pi(u)v - v \mid v \in V, u \in U_H \rangle$. So there exists a non-negative integer k such that

$$\int_{\mathfrak{p}_F^{-k-n}} \pi \begin{pmatrix} 1 & & x\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} (S - q^4)v dx = 0.$$

Since U_H is the center of U , we get

$$(S - q^4) \int_{\mathfrak{p}_F^{-k-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & & x\sqrt{\epsilon} \\ & 1 & \\ & & 1 \end{pmatrix} v dx = 0.$$

It follows from Lemma 3.7, there are $v_1 \in V(n+k-2)$ and $v_2 \in V(n+k-1)$ such that

$$\int_{\mathfrak{p}_F^{-k-n}/\mathfrak{p}_F^{-n}} \pi \begin{pmatrix} 1 & x\sqrt{\epsilon} \\ & 1 \\ & & 1 \end{pmatrix} v dx = \theta'^k v + \eta v_1 + \eta v_2.$$

So we have

$$(S - q^4)(\theta'^k v + \eta v_1 + \eta v_2) = (S - q^4)\theta'^k v = 0.$$

This implies $\theta'^k v \in \eta V(n+k-2)$.

If $k = 0$, then we have $v \in \eta V(n-2)$, as required. Suppose that $k > 0$. Then, by Lemma 3.6, we get $\theta'^{k-1} v \in \eta V(n+k-3)$. By repeating this argument, we conclude that $v \in \eta V(n-2)$. \square

In the remaining of this subsection, we apply Lemma 4.2 to non-generic representations.

Lemma 4.3. *Let (π, V) be an irreducible non-generic representation of G . Then we have $V(n) = \eta V(n-2)$, for any integer n such that $n \geq 2$ and $n > n_\pi$.*

Proof. It suffices to prove that $V(n) \subset \eta V(n-2)$. Due to Proposition 4.1 (ii), we have $V_{U_H}(Z_2) = \{0\}$. For $v \in V(n)$, we see that $p((S - q^4)v) \in V_{U_H}(Z_2) = \{0\}$ since $U/U_H \simeq Z_2$. Thus we get $v \in \eta V(n-2)$ by Lemma 4.2. \square

Theorem 4.4. *If an irreducible non-generic representation (π, V) of G admits a newform, then we have*

$$N_\pi = 0, 1, \text{ or } N_\pi = n_\pi.$$

Proof. Suppose that $N_\pi \geq 2$ and $N_\pi > n_\pi$. Then by Lemma 4.3, we have $V(N_\pi) = \eta V(N_\pi - 2) = \{0\}$. This contradicts the choice of N_π . So we have $N_\pi < 2$ or $N_\pi = n_\pi$. \square

4.2. “Kirillov model”. In the remaining of this section, we assume that (π, V) is an irreducible generic representation of G . For $v \in V$, we define a function φ_v on E^\times by

$$\varphi_v(a) = W_v \begin{pmatrix} a & & \\ & 1 & \\ & & \bar{a}^{-1} \end{pmatrix}, \quad a \in E^\times.$$

Let $\mathcal{C}^\infty(E^\times)$ denote the space of locally constant functions on E^\times . Then we get a map $V \rightarrow \mathcal{C}^\infty(E^\times); v \mapsto \varphi_v$. It is easy to observe that $\langle \pi(u)v - v \mid v \in V, u \in U_H \rangle$ lies in the kernel of this map. We therefore obtain a map $V_{U_H} \rightarrow \mathcal{C}^\infty(E^\times); p(v) \mapsto \varphi_v$. We define an action of P_2 on $\mathcal{C}^\infty(E^\times)$ by

$$\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} \varphi \right) (b) = \psi_E(bx) \varphi(ab), \quad a, b \in E^\times, x \in E.$$

Then the map $V_{U_H} \rightarrow \mathcal{C}^\infty(E^\times); p(v) \mapsto \varphi_v$ is a P_2 -homomorphism.

Lemma 4.5. *For $v \in V$, there exists a non-negative integer m such that $\text{supp } \varphi_v \subset \mathfrak{p}_E^{-m}$.*

Proof. Take a non-negative integer m such that v is fixed by $(1 + M_3(\mathfrak{p}_E^m)) \cap G$. For $a \in E^\times$ and $x \in \mathfrak{p}_E^m$, we have

$$\begin{aligned} \varphi_v(a) &= l(\pi(t(a))v) = l(\pi(t(a)u(x, -x\bar{x}/2))v) = l(\pi(u(ax, -ax\bar{a}\bar{x}/2)t(a))v) \\ &= \psi_E(ax)l(\pi(t(a))v) = \psi_E(ax)\varphi_v(a) \end{aligned}$$

since $\pi(u(x, -x\bar{x}/2))v = v$. This implies that $a\mathfrak{p}_E^m \subset \mathfrak{o}_E$ if $a \in \text{supp } \varphi_v$. So we get $\text{supp } \varphi_v \subset \mathfrak{p}_E^{-m}$. \square

Corollary 4.6. *Let n be a non-negative integer. For any $v \in V(n)$, the function φ_v is \mathfrak{o}_E^\times -invariant and $\text{supp } \varphi_v \subset \mathfrak{o}_E$.*

Proof. The function φ_v is \mathfrak{o}_E^\times -invariant since v is fixed by $t(a)$, $a \in \mathfrak{o}_E^\times$. Applying the proof of Lemma 4.5, we get $\text{supp } \varphi_v \subset \mathfrak{o}_E$. \square

Let $\mathcal{C}_c^\infty(E^\times)$ be the subspace of $\mathcal{C}^\infty(E^\times)$ consisting of the compactly supported functions.

Proposition 4.7. *Let v be an element in V such that $p(v) \in V_{U_H}(Z_2)$. Then $\varphi_v \in \mathcal{C}_c^\infty(E^\times)$.*

Proof. We set $V(U) = \langle \pi(u)v - v \mid v \in V, u \in U \rangle$. Since $V_{U_H}(Z_2) = p(V(U))$, it is enough to prove that $\varphi_{\pi(u)v-v} \in \mathcal{C}_c^\infty(E^\times)$, for $v \in V$ and $u \in U$.

We write $u = u(x, y)$, where $x, y \in E$ such that $y + \bar{y} + x\bar{x} = 0$. Let $a \in E^\times$ such that $\varphi_{\pi(u)v-v}(a) \neq 0$. This implies $(\psi_E(ax) - 1)\varphi_v(a) \neq 0$. So we get $ax \notin \mathfrak{o}_E$. By Lemma 4.5, we conclude that $\text{supp } \varphi_{\pi(u)v-v}$ is compact. \square

By Proposition 4.7, we get a map $V_{U_H}(Z_2) \rightarrow \mathcal{C}_c^\infty(E^\times); p(v) \mapsto \varphi_v$.

Lemma 4.8. *The map $V_{U_H}(Z_2) \rightarrow \mathcal{C}_c^\infty(E^\times); p(v) \mapsto \varphi_v$ is an isomorphism of P_2 -modules.*

Proof. We claim that the map $V_{U_H}(Z_2) \rightarrow \mathcal{C}_c^\infty(E^\times); p(v) \mapsto \varphi_v$ is not zero. Let v be an element in V such that $l(v) \neq 0$. This implies $\varphi_v(1) \neq 0$. Take an element x in E such that $\psi_E(x) \neq 1$ and put $u = u(x, -x\bar{x}/2)$. Then we have $\varphi_{\pi(u)v-v}(1) = (\psi_E(x) - 1)\varphi_v(1) \neq 0$. So we conclude that the map $V_{U_H}(Z_2) \rightarrow \mathcal{C}_c^\infty(E^\times); p(v) \mapsto \varphi_v$ is not zero.

For $f \in \text{ind}_{Z_2}^{P_2}(\psi)$, we define $T(f) \in \mathcal{C}_c^\infty(E^\times)$ by

$$T(f)(a) = f \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, \quad a \in E^\times.$$

Then we get an isomorphism of P_2 -modules $T : \text{ind}_{Z_2}^{P_2}(\psi) \simeq \mathcal{C}_c^\infty(E^\times); f \mapsto T(f)$. Since $V_{U_H}(Z_2) \simeq \text{ind}_{Z_2}^{P_2}(\psi) \simeq \mathcal{C}_c^\infty(E^\times)$ and $\text{ind}_{Z_2}^{P_2}(\psi)$ is irreducible, we see that the map $V_{U_H}(Z_2) \rightarrow \mathcal{C}_c^\infty(E^\times); p(v) \mapsto \varphi_v$ is an isomorphism of P_2 -modules. \square

Now we get a criterion whether a vector in $V(n)$ lies in $\eta V(n-2)$, in terms of Kirillov model.

Lemma 4.9. *Let n be an integer such that $n \geq 2$ and $n > n_\pi$. Suppose that $v \in V(n)$ satisfies $\text{supp } \varphi_v \subset \mathfrak{p}_E$. Then $v \in \eta V(n-2)$.*

Proof. It is easy to check that the assumption implies $\varphi_{(S-q^4)v} = 0$. Since $U/U_H \simeq Z_2$, we have $p((S-q^4)v) \in V_{U_H}(Z_2)$. Thus Lemma 4.8 says that $p((S-q^4)v) = 0$. By Lemma 4.2, we obtain $v \in \eta V(n-2)$. \square

Lemma 4.9 shows the following theorem, which bounds the growth of the dimensions of oldforms.

Theorem 4.10. *Let (π, V) be an irreducible generic representation of G . For any non-negative integer n such that $n+2 > n_\pi$, we have*

$$\dim V(n+2) - \dim V(n) \leq 1.$$

Proof. Since the map $\eta : V(n) \rightarrow V(n+2)$ is injective, it is enough to prove $\dim V(n+2)/\eta V(n) \leq 1$. Let v_1, v_2 be elements in $V(n+2) \setminus \eta V(n)$. Due to Corollary 4.6, the function φ_{v_i} is \mathfrak{o}_E^\times -invariant and $\text{supp } \varphi_{v_i} \subset \mathfrak{o}_E$, for $i = 1, 2$. Put

$$\alpha = \varphi_{v_2}(1), \quad \beta = \varphi_{v_1}(1).$$

By Lemma 4.9, we have $\alpha \neq 0$ and $\beta \neq 0$. Since $\text{supp } \varphi_{\alpha v_1 - \beta v_2} \subset \mathfrak{p}_E$, Lemma 4.9 implies $\alpha v_1 - \beta v_2 \in \eta V(n)$. So we conclude that $\dim V(n+2)/\eta V(n) \leq 1$. \square

Applying Theorem 4.10 to $n = N_\pi - 2$, we obtain the uniqueness of newforms for generic representations whose conductors differ from those of their central characters.

Theorem 4.11. *Suppose that an irreducible generic representation (π, V) of G satisfies $N_\pi > n_\pi$ and $N_\pi \geq 2$. Then we have $\dim V(N_\pi) = 1$.*

Proof. By Theorem 4.10, we have $\dim V(N_\pi) - \dim V(N_\pi - 2) \leq 1$. Since $\dim V(N_\pi) \geq 1$ and $\dim V(N_\pi - 2) = 0$, we get $\dim V(N_\pi) = 1$. \square

We close this section by showing that the newform is a test vector for the Whittaker functional.

Theorem 4.12. *Let π be an irreducible generic representation of G such that $N_\pi \geq 2$ and $N_\pi > n_\pi$. For a non-zero element v in $V(N_\pi)$, we have $W_v(1) \neq 0$.*

Proof. The function φ_v is supported by \mathfrak{o}_E . Since $V(N_\pi - 2) = \{0\}$, it follows from Lemma 4.9 that $\text{supp } \varphi_v \not\subset \mathfrak{p}_E$. Because the function φ_v is \mathfrak{o}_E^\times -invariant, we get $\varphi_v(1) = W_v(1) \neq 0$. \square

5. MAIN THEOREMS

In this section, we prove our main results. We show that the space of newforms is of dimension one. Moreover, we give a formula of the dimensions of the oldforms for generic representations whose conductors are greater than those of their central characters.

5.1. Multiplicity one theorem of newforms. As the well-known fact on K_0 -fixed vectors, we have the following:

Proposition 5.1. *Let (π, V) be an irreducible admissible representation of G . Then we have $\dim V(1) \leq 1$.*

Proof. Let (π, V) be an irreducible admissible representation of G such that $V(1) \neq \{0\}$. It is well known that π can not be supercuspidal. So π can be embedded into a parabolically induced representation $\text{Ind}_B^G \chi$, for some quasi-character χ of T . Let $(\text{Ind}_B^G \chi)^{K_1}$ denote the space of the K_1 -fixed vectors in $\text{Ind}_B^G \chi$. Then we have $\dim V(1) \leq \dim (\text{Ind}_B^G \chi)^{K_1}$. The dimension of $(\text{Ind}_B^G \chi)^{K_1}$ equals to the number of the elements g in $B \backslash G / K_1$ such that χ is trivial on $B \cap gK_1g^{-1}$. So Lemma 2.1 implies $\dim (\text{Ind}_B^G \chi)^{K_1} \leq 1$. This completes the proof. \square

We shall treat the case when Lemma 4.2 is not valid. In this case, we use the Hecke algebra isomorphism established by Moy.

Theorem 5.2. *Let (π, V) be an irreducible admissible representation of G which admits a newform. Suppose that the conductor N_π of π satisfies $N_\pi \geq 2$ and $N_\pi = n_\pi$. Then*

- (i) $\dim V(N_\pi) = 1$;
- (ii) π is not supercuspidal.

Proof. (i) We use the Hecke algebra isomorphism in [7]. Put $n = N_\pi$. We define two open compact subgroups of G by

$$P_{3n-3} = \begin{pmatrix} 1 + \mathfrak{p}_E^{n-1} & \mathfrak{p}_E^{n-1} & \mathfrak{p}_E^{n-1} \\ \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^{n-1} & \mathfrak{p}_E^{n-1} \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^{n-1} \end{pmatrix} \cap G, \quad P_{3n-2} = \begin{pmatrix} 1 + \mathfrak{p}_E^n & \mathfrak{p}_E^{n-1} & \mathfrak{p}_E^{n-1} \\ \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^n & \mathfrak{p}_E^{n-1} \\ \mathfrak{p}_E^n & \mathfrak{p}_E^n & 1 + \mathfrak{p}_E^n \end{pmatrix} \cap G.$$

Since $P_{3n-2} \subset K_n$, all elements in $V(n)$ are fixed by P_{3n-2} . We have $P_{3n-3} = Z_{n-1}(T_H \cap P_{3n-3})P_{3n-2}$ and $T_H \cap P_{3n-3} \subset K_n$. So P_{3n-3} acts on $V(n)$ by an extension ρ of the central character of π .

Let ψ_F be a non-trivial additive character of F with conductor \mathfrak{p}_F . For an element $\beta \in M_3(E)$, we define a map $\psi_\beta : M_3(E) \rightarrow \mathbf{C}^\times$ by

$$\psi_\beta(x) = \psi_F(\text{tr}_{E/F} \circ \text{tr}(\beta(x-1))), \quad x \in M_3(E).$$

By [6] Theorem 2.13, there exists

$$\beta = \varpi^{1-n} \begin{pmatrix} A & 0 & 0 \\ 0 & a\sqrt{\epsilon} & 0 \\ 0 & 0 & -\overline{A} \end{pmatrix}, \quad A \in \mathfrak{o}_E/\mathfrak{p}_E, \quad a \in \mathfrak{o}_F/\mathfrak{p}_F$$

such that

$$\rho(p) = \psi_\beta(p), \quad p \in P_{3n-3}.$$

Because ρ is trivial on $T_H \cap P_{3n-3}$, we have $A \in \mathfrak{p}_E$. So we may assume $A = 0$. Since we are assuming $n = n_\pi$, we obtain $a \in \mathfrak{o}_F^\times$.

We define an open compact subgroup J of G as in [7] p. 191. If $n = 2m$, we put

$$J = \begin{pmatrix} 1 + \mathfrak{p}_E^{n-1} & \mathfrak{p}_E^m & \mathfrak{p}_E^{n-1} \\ \mathfrak{p}_E^m & 1 + \mathfrak{p}_E^{n-1} & \mathfrak{p}_E^m \\ \mathfrak{p}_E^n & \mathfrak{p}_E^m & 1 + \mathfrak{p}_E^{n-1} \end{pmatrix} \cap G.$$

If $n = 2m + 1$, we set

$$J = \begin{pmatrix} 1 + \mathfrak{p}_E^{n-1} & \mathfrak{p}_E^m & \mathfrak{p}_E^{n-1} \\ \mathfrak{p}_E^{m+1} & 1 + \mathfrak{p}_E^{n-1} & \mathfrak{p}_E^m \\ \mathfrak{p}_E^n & \mathfrak{p}_E^{m+1} & 1 + \mathfrak{p}_E^{n-1} \end{pmatrix} \cap G.$$

Then J contains P_{3n-3} . We can extend ρ to a character of J which is trivial outside of P_{3n-3} , which is also denoted by ρ . We put $G' = HZ$, $J' = J \cap G'$ and $\rho' = \rho|_{J'}$. If we denote by $\mathcal{H}(G//J, \rho)$, $\mathcal{H}(G'//J', \rho')$ the Hecke algebras of G , G' associated to (J, ρ) , (J', ρ') respectively, then by [7] Corollary 2.8, there exists a support-preserving algebra isomorphism

$$(5.3) \quad i : \mathcal{H}(G//J, \rho) \simeq \mathcal{H}(G'//J', \rho').$$

The ρ -isotypic component π^ρ of π is an irreducible $\mathcal{H}(G//J, \rho)$ -module. There is a left ideal I of $\mathcal{H}(G//J, \rho)$ such that $\pi^\rho \simeq \mathcal{H}(G//J, \rho)/I$. By (5.3), there is an irreducible admissible representation τ of G' such that $\tau^{\rho'} \simeq \mathcal{H}(G'//J', \rho')/i(I)$. Since $G' = ZH$, we can view τ as an irreducible admissible representation of $H \simeq \mathrm{U}(1, 1)(E/F)$. So there is an isomorphism $i : \pi^\rho \simeq \tau^{\rho'}$ such that

$$i(\pi(f)v) = \tau(i(f))i(v), \quad v \in \pi^\rho, \quad f \in \mathcal{H}(G//J, \rho).$$

Replacing J with $\eta^m J \eta^{-m}$, we may assume $J \subset K_n Z_{n-1}$. Then one can observe that $V(n) \subset \pi^\rho$. Let f_ρ be the identity element in $\mathcal{H}(G//J, \rho)$. For $g \in G$, we denote by δ_g the Dirac point mass at g . Then we have

$$(5.4) \quad \pi(f_\rho * \delta_k * f_\rho)v = \pi(f_\rho)\pi(k)\pi(f_\rho)v = v, \quad v \in V(n), \quad k \in K_n \cap H.$$

Let $V'(n)$ be the image of $V(n)$ in $\tau^{\rho'}$. We denote by e_ρ the identity element in $\mathcal{H}(G'//J', \rho')$. By (5.4), we get

$$\tau(e_\rho * \delta_k * e_\rho)v' = v', \quad v' \in V'(n), \quad k \in K_n \cap H.$$

So we obtain

$$\begin{aligned} \tau(e_\rho) \int_{K_n \cap H} \tau(k)v' dk &= \int_{K_n \cap H} \tau(e_\rho)\tau(k)\tau(e_\rho)v' dk \\ &= \int_{K_n \cap H} \tau(e_\rho * \delta_k * e_\rho)v' dk \\ &= \mathrm{vol}(K_n \cap H)v', \end{aligned}$$

for $v' \in V'(n)$. This implies

$$V'(n) \subset \tau(e_\rho)\tau^{K_n \cap H},$$

where $\tau^{K_n \cap H}$ is the space of $K_n \cap H$ -fixed vectors in τ . Since $K_n \cap H$ is a good maximal compact subgroup of H , we obtain $\dim V(n) = \dim V'(n) \leq \dim \tau^{K_n \cap H} \leq 1$.

(ii) We have seen that τ has a non-zero $K_n \cap H$ -fixed vector. Since $K_n \cap H$ is a good maximal compact subgroup of H , τ can not be supercuspidal. As remarked in [7] p. 195, this implies that π is not supercuspidal. \square

Let (π, V) be an irreducible supercuspidal representation of G . It is well known that $\dim V(0) = \dim V(1) = 0$. Therefore we obtain the following

Corollary 5.5. *Let π be an irreducible supercuspidal representation of G .*

- (i) *Suppose that π is generic. Then we have $N_\pi \geq 2$ and $N_\pi > n_\pi$.*
- (ii) *If π is not generic, then π has no K_n -fixed vectors for all $n \geq 0$.*

Proof. Part (i) follows from Theorem 5.2 (ii). Suppose that π is non-generic and admits a newform. Then, by Theorem 5.2 (ii), we have $N_\pi \geq 2$ and $N_\pi > n_\pi$. This contradicts Theorem 4.4. So if π is not generic, then π has no K_n -fixed vectors for all $n \geq 0$. This completes the proof of (ii). \square

We shall prove our main theorem.

Theorem 5.6. *Let (π, V) be an irreducible admissible representation of G which admits a newform. Then the space $V(N_\pi)$ of newforms for π is one-dimensional.*

Proof. Recall that K_0 is a good maximal compact subgroup of G . It is well known that $\dim V(0) \leq 1$. Due to Proposition 5.1, we have $\dim V(1) \leq 1$. So we may assume that $N_\pi \geq 2$.

If we further suppose that $N_\pi = n_\pi$, then the assertion follows from Theorem 5.2 (i). Suppose that $N_\pi > n_\pi$ and $N_\pi \geq 2$. Then Theorem 4.4 says that π should be generic. So we get $\dim V(N_\pi) = 1$ by Theorem 4.11. This completes the proof. \square

5.2. Oldforms for generic representations. The following theorem bounds the dimensions of oldforms for generic representations.

Proposition 5.7. *Let (π, V) be an irreducible generic representation of G . Then we have*

$$\dim V(n) \leq \left\lfloor \frac{n - N_\pi}{2} \right\rfloor + 1,$$

for $n \geq N_\pi$.

Proof. By Theorem 5.6, we have $\dim V(N_\pi) = 1$. We claim that $\dim V(N_\pi + 1) \leq 1$. If $N_\pi = 0$, we have $\dim V(N_\pi + 1) \leq 1$ by Proposition 5.1. Suppose that $N_\pi \geq 1$. Then, by Theorem 4.10, we obtain $\dim V(N_\pi + 1) - \dim V(N_\pi - 1) \leq 1$. So we get $\dim V(N_\pi + 1) \leq 1$.

For $\delta \in \{0, 1\}$ and $k \in \mathbf{Z}_{\geq 0}$, we have $\dim V(N_\pi + \delta + 2k) \leq k + \dim V(N_\pi + \delta) \leq k + 1$, by Theorem 4.10. This completes the proof. \square

We give a basis for oldforms for generic representations π which satisfy $N_\pi \geq 2$ and $N_\pi > n_\pi$.

Theorem 5.8. *Let (π, V) be an irreducible generic representation of G . Suppose that $W_v(1) \neq 0$ for all non-zero elements v in $V(N_\pi)$. Then, for $n \geq N_\pi$, the set $\{\theta'^i \eta^j v \mid i + 2j + N_\pi = n\}$ forms a basis for $V(n)$. In particular,*

$$\dim V(n) = \left\lfloor \frac{n - N_\pi}{2} \right\rfloor + 1.$$

Proof. By Proposition 5.7, it is enough to prove that $\{\theta'^i \eta^j v \mid i + 2j + N_\pi = n\}$ is linearly independent. Let m be a non-negative integer. For $v \in V(m)$, we have

$$\varphi_{\eta v}(1) = \varphi_v(\varpi^{-1}), \quad \varphi_{\theta' v}(1) = \varphi_v(\varpi^{-1}) + q\varphi_v(1),$$

by Proposition 3.3. Due to Corollary 4.6, we obtain

$$\varphi_{\eta v}(1) = 0, \quad \varphi_{\theta' v}(1) = q\varphi_v(1).$$

So for $i, j \geq 0$, we get

$$\varphi_{\theta'^i \eta^j v}(1) = \begin{cases} q^i \varphi_v(1), & \text{if } j = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Let v be a non-zero element in $V(N_\pi)$. Suppose that $\sum_{i+2j+N_\pi=n} \alpha_j \theta'^i \eta^j v = 0$ ($\alpha_j \in \mathbf{C}$). Then we have

$$0 = \varphi_{\sum_j \alpha_j \theta'^i \eta^j v}(1) = \alpha_0 q^{n-N_\pi} \varphi_v(1),$$

so that $\alpha_0 = 0$ by assumption. So we get $\sum_{i+2j+N_\pi=n, j \geq 1} \alpha_j \theta'^i \eta^j v = 0$. Since η is injective and commutes with θ' , we have $\sum_{i+2j+N_\pi=n, j \geq 1} \alpha_j \theta'^i \eta^{j-1} v = 0$. Repeating this argument, we obtain $\alpha_j = 0$, for all j . \square

Remark 5.9. Suppose that an irreducible generic representation π of G satisfies $N_\pi \geq 2$ and $N_\pi > n_\pi$. Then Theorem 4.12 says that the assumption of Theorem 5.8 holds for π .

5.3. Oldforms for non-generic representations. We close this paper with a result on the possibilities of the dimensions of oldforms for non-generic representations.

Theorem 5.10. *Let (π, V) be an irreducible non-generic representation of G which admits a newform. For any non-negative integer k , the following holds.*

- (i) $\dim V(N_\pi + 2k) = 1$.
- (ii) If $N_\pi \geq 1$, then $\dim V(N_\pi + 2k + 1) = 0$.
- (iii) Suppose that $N_\pi = 0$. Then $\dim V(2k + 1) = \dim V(1)$.

Proof. (i) Lemma 4.3 implies $V(N_\pi + 2k) = \eta^k V(N_\pi)$. So the assertion follows from Theorem 5.6. (ii) Suppose that $N_\pi \geq 1$. Lemma 4.3 says that $V(N_\pi + 2k + 1) = \eta^{k+1} V(N_\pi - 1) = \{0\}$. (iii) By Lemma 4.3, we have $V(2k + 1) = \eta^k V(1)$. \square

REFERENCES

- [1] E. M. Baruch. On the gamma factors attached to representations of $U(2, 1)$ over a p -adic field. *Israel J. Math.*, 102:317–345, 1997.
- [2] I. N. Bernštein and A. V. Zelevinskii. Representations of the group $GL(n, F)$, where F is a local non-Archimedean field. *Uspehi Mat. Nauk*, 31(3(189)):5–70, 1976.
- [3] W. Casselman. On some results of Atkin and Lehner. *Math. Ann.*, 201:301–314, 1973.
- [4] H. Jacquet, I. Piatetski-Shapiro, and J. Shalika. Conducteur des représentations du groupe linéaire. *Math. Ann.*, 256(2):199–214, 1981.
- [5] J. Lansky and A. Raghuram. Conductors and newforms for $U(1, 1)$. *Proc. Indian Acad. Sci. (Math Sci.)*, 114(4):319–343, 2004.
- [6] L. Morris. Tamely ramified supercuspidal representations of classical groups. I. Filtrations. *Ann. Sci. École Norm. Sup. (4)*, 24:705–738, 1991.
- [7] A. Moy. Representations of $U(2, 1)$ over a p -adic field. *J. Reine Angew. Math.*, 372:178–208, 1986.
- [8] M. Reeder. Old forms on GL_n . *Amer. J. Math.*, 113(5):911–930, 1991.
- [9] B. Roberts and R. Schmidt. *Local newforms for $GSp(4)$* , volume 1918 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, KYOTO UNIVERSITY, OIWAKE KITA-SHIRAKAWA SAKYO KYOTO 606-8502 JAPAN

E-mail address: miyauchi@math.kyoto-u.ac.jp